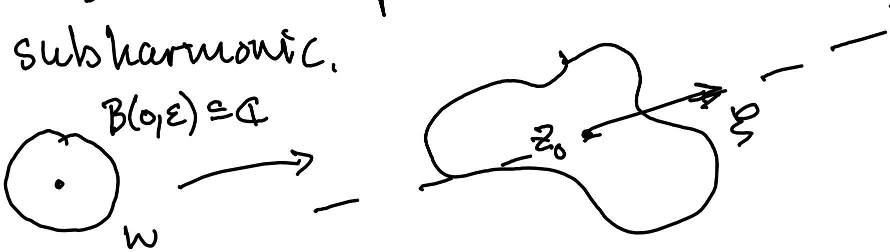
Lecture 7

Pseudoconvexity & Plurisubharmonicity

The notion of being a domain of holomorphy is, loosely, the property that there is no boundary point where all holom. fcns extend across to be holom. in a nbhd of that point. But the def. and subseq. characterization are both global in the sense that you cannot check DoH prop. locally near each boundary point. We would like such a characterization.

<u>Def.</u> A function $u : \Omega \to [-\infty, \infty)$ is <u>plurisubharmonic</u> (PSH) if u is upper semicont. and for all $z \in \Omega$ and $\xi \in \mathbb{C}^n$ the function $w \to u(z + w\xi)$ is subharmonic.



Subharmonic functions in $\mathbb{C}$. (SH)

If $\Omega \subseteq \mathbb{C}$, $u : \Omega \to [-\infty, \infty)$ is subharmonic if u is upper semicont. (USC) and, for all open disks D s.t. $\overline{D} \subseteq \Omega$ and all polynomials $p(z) = \sum_{k=0}^{m} a_k z^k$ s.t.

$u \leq \operatorname{Re} p$ on $\partial D \Rightarrow u \leq \operatorname{Re} p$ in D.

Subharmonic functions have the sub mean value property: If $D = B(a, r)$ is a disk s.t. $\overline{D} \subseteq \Omega$ and μ is a probability measure on $[0, r]$, then

$$u(a) \leq \frac{1}{2\pi} \int_0^r \int_0^{2\pi} u(a + se^{it})\, dt\, d\mu(s)$$

Rem. This characterization is "$\Leftrightarrow$".

If $u \in \mathcal{C}^2$, then u is SH and $\Delta u \geq 0$.

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4 \frac{\partial^2}{\partial z \partial \bar{z}}$$

Prop 1. If $u: \Omega \to \mathbb{R}$ is C^2, then

$$u \in PSH(\Omega) \iff \sum_{k,l} \frac{\partial^2 u}{\partial z_k \partial \bar{z}_l}(z)\, \xi_k \bar{\xi}_l \geq 0 \quad (\forall \xi \in \mathbb{C}^n).$$

Important Example. Let $f \in \mathcal{O}(\Omega)$. Then, $u(z) = \log|f(z)|$ is $PSH(\Omega)$. This follows immediately from Max Mod Principle. Note that this u does not satisfy the regularity condition in Prop 1, since $u(z) = -\infty$ where $f(z) = 0$.

Thm 1. Let Ω be DoH. Then, $u(z) = -\log d(z, \Omega^c)$ is $PSH(\Omega)$.

For pf, we need a strengthening of the equivalent condition (ii) in DoH result.

<u>Prop 2</u> Ω DoH $\Leftrightarrow$ $\forall K \subset\subset \Omega$ and $f \in \mathcal{O}(\Omega)$:

$$\sup_{z\in K} \frac{|f(z)|}{d(z,\Omega^c)} = \sup_{z\in \hat{K}_\Omega} \frac{|f(z)|}{d(z,\Omega^c)}.$$

We argued that the case where $f(z) =$ constant, i.e.

$$\inf_{z\in K} d(z,\Omega^c) = \inf_{z\in \hat{K}_\Omega} d(z,\Omega^c)$$

follows from what we proved concerning "distance to Ω^c" defined using scaling of a fixed polydisk. The more general result Prop 2 follows in a similar manner from this. See Hörmander Sec. 2.5.

We shall prove Thm 1 using Prop 2.

Pf of Thm 1. Pick $z_0 \in \Omega$, $\xi \in \mathbb{C}^n$, and $r>0$ s.t. $\{z_0 + w\xi : |w| \le r\} \subseteq \Omega$. Let $D = \{w \in \mathbb{C} : |w| < r\}$ and $v(w) = u(z_0 + w\xi)$, $u(z) = -\log d(z, \Omega^c)$. WTS if $p(w) = \sum_{k=0}^{m} a_k w^k$ a holom. polynomial s.t. $v(w) \le \operatorname{Re} p(w)$ on ∂D, then $v \le \operatorname{Re} p$ in D. Now,

$$v \le \operatorname{Re} p \iff d(z, \Omega^c) \ge |e^{-Q(z)}| \iff \frac{|e^{-Q}|}{d(\cdot, \Omega^c)} \le 1$$

for $z = z_0 + w\xi$, if $Q(z)$ is a holom. poly. s.t. $Q(z_0 + w\xi) = p(w)$ (such $Q(z)$ are easily seen to exist). Let $F = e^{-Q} \in \mathcal{O}(\mathbb{C}^n) \subseteq \mathcal{O}(\Omega)$ and $K = \{z_0 + w\xi : |w| = 1\} \subset\subset \Omega$.

Claim. $A = \{z_0 + w\xi : |w| \le 1\} \subseteq \hat{K}_\Omega$.

The claim follows from Max Mod Princ.: If $f \in \mathcal{O}(\Omega)$, then $h(w) = f(z_0 + w\xi)$ is holom. in D and continuous in $\overline{D}$.

Thus, $\sup_{\overline{D}} |h| = \sup_{\partial D} |h| = \sup_K |f| \Rightarrow$

for any $z_0 + w\xi$, $w \in D$, $|f(z_0 + \xi w)| \leq \sup_K |f|$

$\Rightarrow z_0 + w\xi \in \hat{K}_\Omega$, as claimed.

Now, by Prop 2, for any $w \in D \Rightarrow$

$$\frac{|F(w)|}{d(w, \Omega^c)} \leq \sup_K \frac{|F|}{d(\cdot, \Omega^c)} \leq 1$$

By assumption $v \leq \operatorname{Re} p$ on ∂D

$\Rightarrow$

$$v(w) \leq \operatorname{Re} p(w). \qquad \square$$

<u>Def.</u> $\Omega \subseteq \mathbb{C}^n$ is <u>pseudoconvex</u> (ψcvx) if $-\log d(z, \Omega^c)$ is $PSH(\Omega)$.

Thus, Thm 1 can be phased as

Ω DoH $\Rightarrow$ Ω ψcvx.

It actually holds that DoH $\Leftrightarrow$ ψcvx. The converse is harder and was known as the Levi problem until it was resolved in the 50's independently by Oka, Bremerman, Norguet. The solution consists of showing that the $\bar\partial$-eq. can be solved in $C^\infty_{(0,q)}(\Omega)$ when Ω is ψcvx and then that such solvability $\Rightarrow$ Ω is DoH. We do not have the time to develop this theory. We simply state

<u>Thm 2.</u> Ω ψcvx $\Leftrightarrow$ Ω DoH.

Ch. 4 in Hörmander is dedicated to solvability of $\bar\partial$ in Ω.